2-Ruled Calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$

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1 Introduction

In this paper we study certain *calibrated* 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$, which are known as *coassociative* in $\mathbb{R}^7$, and are called *special Lagrangian* (SL) and *Cayley* in $\mathbb{R}^8$. We introduce the notion of *2-ruled* 4-folds in $\mathbb{R}^n$; that is, submanifolds M of $\mathbb{R}^n$ admitting a fibration $\pi : M \to \Sigma$ over some 2-fold Σ such that each fibre, $\pi^{-1}(\sigma)$ for $\sigma \in \Sigma$, is an affine 2-plane in $\mathbb{R}^n$. We say that M is *r-framed* if we are given an oriented basis for each fibre in a smooth manner. In such circumstances there exist orthogonal smooth maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^{n-1}$ and a smooth map $\psi : \Sigma \to \mathbb{R}^n$ such that

$$M = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\ r_1, r_2 \in \mathbb{R}\}.$$

Then the *asymptotic cone* M_0 of M is given by:

$$M_0 = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) : \sigma \in \Sigma,\ r_1, r_2 \in \mathbb{R}\}.$$

The motivation for this paper comes from the study of *ruled* SL 3-folds in $\mathbb{C}^3$ in [5] and ruled associative 3-folds in $\mathbb{R}^7$ in [6]: these are calibrated 3-folds that are fibred over a 2-fold by (real) affine straight lines.

We begin in §2 by discussing calibrated geometry in $\mathbb{R}^7$ and $\mathbb{R}^8$. In particular, we show that coassociative and SL 4-folds can be considered as special cases of Cayley 4-folds. In §3 we give the definitions required to study 2-ruled submanifolds.

In §4 we give our main result, Theorem 4.6, which is on non-planar, r-framed, 2-ruled Cayley 4-folds. This result characterizes the Cayley condition in terms of a coupled system of nonlinear first-order partial differential equations that ϕ_1 and ϕ_2 satisfy, and another such equation on ψ which is *linear* in ψ. Therefore, for a fixed non-planar, r-framed, 2-ruled Cayley cone M_0, the space of r-framed 2-ruled Cayley 4-folds M which have asymptotic cone M_0 has the structure of a finite-dimensional vector space.

Theorem 4.10 gives a means of constructing 2-ruled Cayley 4-folds M from a 2-ruled Cayley cone M_0, satisfying a certain condition, involving *holomorphic vector fields*. Using Theorem 4.6 and Theorem 4.10, we deduce corresponding results for SL and coassociative 4-folds.

Finally, in §5, we give explicit examples of 2-ruled 4-folds. We use Theorem 4.10 to construct U(1)-invariant 2-ruled Cayley 4-folds from a $\mathrm{U}(1)^3$-invariant 2-ruled Cayley cone. Our other examples are based on ruled 3-folds and complex cones.

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2 Calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$

We begin by defining the basic concepts of *calibrated geometry* following the approach in [3]. Manifolds are assumed to be smooth and nonsingular everywhere unless stated otherwise and submanifolds are considered to be immersed.

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is an oriented k-dimensional vector subspace V of T_xM, for some x in M. Given an oriented tangent k-plane V on M, $g|_V$ is a Euclidean metric on V and hence, using $g|_V$ and the given orientation on V, we have a natural volume form vol_V on V which is a k-form on V.

Let η be a closed k-form on M. Then η is a *calibration* on M if $\eta|_V \leq \mathrm{vol}_V$ for all oriented tangent k-planes V on M, where $\eta|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and so $\eta|_V \leq \mathrm{vol}_V$ if $\alpha \leq 1$. Let N be an oriented k-dimensional submanifold of M. Then T_xN is an oriented tangent k-plane for all $x \in N$. We say that N is a *calibrated submanifold* or *η-submanifold* if $\eta|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Calibrated submanifolds are *minimal* submanifolds [3, Theorem II.4.2]. We define calibrations on $\mathbb{R}^7$ and $\mathbb{R}^8$ as in [4, Chapter X].

Definition 2.2 Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 3-form φ by:

$$\varphi = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}. \tag{1}$$

The 4-form $*\varphi$, where φ and $*\varphi$ are related by the Hodge star, is given by:

$$*\varphi = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}. \tag{2}$$

By [3, Theorem IV.1.16], $*\varphi$ is a calibration on $\mathbb{R}^7$ and submanifolds calibrated with respect to $*\varphi$ are called *coassociative* 4-folds.

The subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving φ is G_2. It is a compact, connected, simply connected, simple, 14-dimensional Lie group, which also preserves the Euclidean metric on $\mathbb{R}^7$, the orientation on $\mathbb{R}^7$ and $*\varphi$. We note that, by [3, Theorem IV.1.4], φ is a calibration on $\mathbb{R}^7$ and φ-submanifolds are called *associative* 3-folds.

Definition 2.3 Let $(x_1,\ldots,x_8)$ be coordinates on $\mathbb{R}^8$. Define a 4-form Φ by:

$$\begin{aligned}\Phi = {} & d\mathbf{x}_{1234}+d\mathbf{x}_{1256}+d\mathbf{x}_{1278}+d\mathbf{x}_{1357}-d\mathbf{x}_{1368}-d\mathbf{x}_{1458}-d\mathbf{x}_{1467}\\ & +d\mathbf{x}_{5678}+d\mathbf{x}_{3478}+d\mathbf{x}_{3456}+d\mathbf{x}_{2468}-d\mathbf{x}_{2457}-d\mathbf{x}_{2367}-d\mathbf{x}_{2358}.\end{aligned} \qquad (3)$$

By [3, Theorem IV.1.24], Φ is a calibration on $\mathbb{R}^8$ and submanifolds calibrated with respect to Φ are called *Cayley* 4-folds.

The subgroup of $\mathrm{GL}(8,\mathbb{R})$ preserving Φ is $\mathrm{Spin}(7)$. It is a compact, connected, simply connected, simple, 21-dimensional Lie group, which preserves the Euclidean metric and the orientation on $\mathbb{R}^8$. It is isomorphic to the double cover of $\mathrm{SO}(7)$.

It is worth noting that our formulae (1)-(3) are in agreement with [1] and [4] but differ from those given in [3]. However, they are equivalent up to a coordinate transformation and a possible change of sign, relating to a reversal of orientation of the calibrated submanifold, so the results from [3] may be applied. Certain formulae used later from [3] then need slight modification in order to be consistent with this choice of representation. These changes shall be pointed out to the reader.

We may consider $\mathbb{R}^8$ as $\mathbb{C}^4$ with complex coordinates $z_1=x_1+ix_2$, $z_2=x_3+ix_4$, $z_3=x_5+ix_6$, $z_4=x_7+ix_8$, so we define a calibration on $\mathbb{C}^4$ which can easily be generalised to $\mathbb{C}^m$.

Definition 2.4 Let $\mathbb{C}^4$ have complex coordinates (z_1,z_2,z_3,z_4) and metric $g=|dz_1|^2+\ldots+|dz_4|^2$. Define a real 2-form ω and a complex 4-form Ω on $\mathbb{C}^4$ by:

$$\omega=\frac{i}{2}(dz_1\wedge d\bar{z}_1+\ldots+dz_4\wedge d\bar{z}_4), \quad \Omega=dz_1\wedge dz_2\wedge dz_3\wedge dz_4. \qquad (4)$$

Let L be a real oriented 4-fold in $\mathbb{C}^4$. Then L is a *special Lagrangian* (SL) 4-fold in $\mathbb{C}^4$ with *phase* $e^{i\theta}$ if L is calibrated with respect to $\cos\theta\,\mathrm{Re}\,\Omega+\sin\theta\,\mathrm{Im}\,\Omega$. If the phase of L is unspecified it is taken to be one.

Alternative characterizations of these calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$ are given in [3]. The first follows from [3, Proposition IV.4.5 & Theorem IV.4.6].

Proposition 2.5 *Let M be a 4-fold in $\mathbb{R}^7$. Then M, with an appropriate orientation, is coassociative if and only if $\varphi|_M \equiv 0$.*

The second, for SL 4-folds, is taken from [3, Corollary III.1.11].

Proposition 2.6 *Let L be a real 4-fold in $\mathbb{C}^4$. Then L, with the correct orientation, is an SL 4-fold in $\mathbb{C}^4$ with phase $e^{i\theta}$ if and only if $\omega|_L \equiv 0$ and $(\sin\theta \operatorname{Re}\Omega - \cos\theta \operatorname{Im}\Omega)|_L \equiv 0$.*

The final result is taken from [3, Corollary IV.1.29]. It requires the definition of the *fourfold cross product* of four vectors in $\mathbb{R}^8$, for which we identify $\mathbb{R}^8$ with the *octonions*, or *Cayley numbers*, $\mathbb{O}$.

Definition 2.7 Let $x, y, z, w \in \mathbb{O} \cong \mathbb{R}^8$. We define the *triple cross product* of x, y, z by:

$$x \times y \times z = -\frac{1}{2}\left(x(\bar{y}z) - z(\bar{y}x)\right). \tag{5}$$

The *fourfold cross product* of x, y, z, w is given by:

$$x \times y \times z \times w = \frac{1}{4}\big(\bar{x}(y \times z \times w) + \bar{y}(z \times x \times w) + \bar{z}(x \times y \times w) + \bar{w}(y \times x \times z)\big).$$

Note that $x \times y \times z \times w$ is an alternating multi-linear form and that these definitions differ in sign from those given in [3] because of the choice of Φ in (3).

Proposition 2.8 *Let V be a 4-plane in $\mathbb{R}^8$ with basis (x, y, z, w). Then V, with an appropriate orientation, is Cayley if and only if $\operatorname{Im}(x \times y \times z \times w) = 0$.*

In §4 we shall need the following properties of Cayley 4-folds that relate to *real analyticity*. The first is a consequence of the minimality of Cayley 4-folds, as discussed in [3], and the second is taken from [3, Theorem IV.4.3].

Theorem 2.9 *Suppose that M is a Cayley 4-fold in $\mathbb{R}^8$. Then M is real analytic.*

Theorem 2.10 *Let N be a real analytic 3-fold in $\mathbb{R}^8$. Then there exists a unique real analytic Cayley 4-fold in $\mathbb{R}^8$ containing N.*

We end the section by giving the following result, which shows that coassociative and SL 4-folds are special cases of Cayley 4-folds in $\mathbb{R}^8$, the proof of which is immediate from equations (1)-(4).

Proposition 2.11 *If we consider $\mathbb{R}^8 \cong \mathbb{R} \oplus \mathbb{R}^7$, with x_1 as the coordinate on $\mathbb{R}$ and coordinates on $\mathbb{R}^7$ labelled as $(x_2, \ldots, x_8)$, then*

$$\Phi = dx_1 \wedge \varphi + *\varphi. \tag{6}$$

Hence, if $M \subseteq \mathbb{R}^7 \subseteq \mathbb{R}^8$ is a Cayley 4-fold, then M is a coassociative 4-fold. Conversely, let N be a coassociative 4-fold and $M = \{0\} \times N \subseteq \mathbb{R} \oplus \mathbb{R}^7 = \mathbb{R}^8$. Then M is a Cayley 4-fold.

Consider $\mathbb{R}^8 \cong \mathbb{C}^4$ with $(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$ as complex coordinates. Then

$$\Phi = \frac{1}{2}\,\omega \wedge \omega + \mathrm{Re}\ \Omega. \tag{7}$$

Hence, if $M \subseteq \mathbb{C}^4 \cong \mathbb{R}^8$ is a Cayley 4-fold such that $\omega|_M \equiv 0$, then M is an SL 4-fold in $\mathbb{C}^4$. Conversely, if L is an SL 4-fold in $\mathbb{C}^4 \cong \mathbb{R}^8$, then L is a Cayley 4-fold.

3 2-Ruled 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$

We begin by defining *2-ruled* 4-folds in $\mathbb{R}^n$. We take a *cone* to be a submanifold of $\mathbb{R}^n$ which is invariant under dilations and nonsingular except possibly at 0.

Definition 3.1 Let M be a 4-fold in $\mathbb{R}^n$. A *2-ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that $\pi^{-1}(\sigma)$ is an affine 2-plane in $\mathbb{R}^n$ for all $\sigma \in \Sigma$. The triple (M, Σ, π) is a *2-ruled* 4-fold in $\mathbb{R}^n$.

An *r-framing* for a 2-ruling (Σ, π) of M is a choice of oriented orthonormal basis, or frame, for the affine 2-plane $\pi^{-1}(\sigma)$ given by the 2-ruling, for each $\sigma \in \Sigma$, which varies smoothly with σ. Then (M, Σ, π) with an r-framing is called *r-framed*.

Let (M, Σ, π) be an r-framed 2-ruled 4-fold in $\mathbb{R}^n$. For each $\sigma \in \Sigma$, define $(\phi_1(\sigma), \phi_2(\sigma))$ to be the oriented orthonormal basis for $\pi^{-1}(\sigma)$ given by the r-framing. Then $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^{n-1}$ are smooth maps. Define $\psi : \Sigma \to \mathbb{R}^n$ such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi_1(\sigma)$ and $\phi_2(\sigma)$. Then ψ is a smooth map and

$$M = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\ r_1, r_2 \in \mathbb{R}\}. \tag{8}$$

We define the *asymptotic cone* M_0 of a 2-ruled 4-fold M as the set of points in planes Π including the origin such that Π is parallel to $\pi^{-1}(\sigma)$ for some $\sigma \in \Sigma$. If M is r-framed, then

$$M_0 = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) : \sigma \in \Sigma,\ r_1, r_2 \in \mathbb{R}\} \tag{9}$$

and is usually a 4-dimensional cone; that is, whenever the map $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^{n-1}$ given by $\iota(\sigma, e^{i\theta}) = \cos\theta\phi_1(\sigma) + \sin\theta\phi_2(\sigma)$ is an immersion.

Let (M, Σ, π) be a 2-ruled 4-fold in $\mathbb{R}^n$. Let

$$P = \{(\mathbf{v}, \sigma) \in \mathcal{S}^{n-1} \times \Sigma : \mathbf{v} \text{ is a unit vector parallel to } \pi^{-1}(\sigma),\, \sigma \in \Sigma\}$$

and let $\pi_P : P \to \Sigma$ be given by $\pi_P(\mathbf{v}, \sigma) = \sigma$. Clearly, $\pi_P : P \to \Sigma$ is an $\mathcal{S}^1$ bundle over Σ. Note that (M, Σ, π) admits an r-framing if and only if this bundle is trivializable. Therefore, if M is orientable and Σ is non-orientable, e.g. $\Sigma \cong \mathcal{K}$ where $\mathcal{K}$ is the Klein bottle, then a 2-ruling (Σ, π) cannot be r-framed. Moreover, if M is r-framed then M_0 is not necessarily 4-dimensional. For example, if we take $\Sigma = \mathbb{R}^2$ and define ϕ_1, ϕ_2, ψ by $\phi_1(x, y) = (1, 0, 0, 0)$, $\phi_2(x, y) = (0, 1, 0, 0)$ and $\psi(x, y) = (0, 0, x, y)$ for $x, y \in \mathbb{R}$, then M, as defined by (8), is an r-framed 2-ruled 4-fold since $M = \mathbb{R}^4$, but $M_0 = \mathbb{R}^2$. We also note that any r-framed 2-ruled 4-fold is defined by three maps ϕ_1, ϕ_2, ψ as in (8). We may thus construct 2-ruled calibrated 4-folds by formulating *evolution equations* for ϕ_1, ϕ_2, ψ.

We justify the terminology of asymptotic cone as given in Definition 3.1. To do this we define the term *asymptotically conical with order* $O(r^\alpha)$, where r is the radius function on $\mathbb{R}^n$, as in [5, Definition 3.5].

Definition 3.2 Let M be a closed submanifold of $\mathbb{R}^n$ and let M_0 be a closed cone in $\mathbb{R}^n$. Then M is *asymptotically conical to* M_0 *with order* $O(r^\alpha)$, for some $\alpha < 1$, if there exist $R > 0$, a compact subset K of M and a diffeomorphism $\Psi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$|\nabla^k(\Psi(\mathbf{x}) - I(\mathbf{x}))| = O(r^{\alpha-k}) \quad \text{for } k = 0, 1, 2, \dots \text{ as } r \to \infty, \tag{10}$$

where $\bar{B}_R$ is the closed ball of radius R in $\mathbb{R}^n$ and $I : M_0 \setminus \bar{B}_R \to \mathbb{R}^n$ is the inclusion map. Here $|\,.\,|$ is calculated using the cone metric on $M_0 \setminus \bar{B}_R$, and ∇ is a combination of the Levi-Civita connection derived from the cone metric and the flat connection on $\mathbb{R}^n$, which acts as partial differentiation.

Let M be an r-framed 2-ruled 4-fold with asymptotic cone M_0. Writing M and M_0 in the form (8) and (9) respectively, we define, for some $R > 0$ and compact $K \subseteq M$, a diffeomorphism $\Psi : M_0 \setminus \bar{B}_R \to M \setminus K$ by $\Psi(r_1\phi_1(\sigma) + r_2\phi_2(\sigma)) = r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma)$ for all $\sigma \in \Sigma$ and $|r_1|^2 + |r_2|^2 > R^2$. Then, if Σ is compact so that ψ is bounded, Ψ satisfies (10) for $\alpha = 0$. Therefore M is asymptotically conical to M_0 with order $O(1)$.

4 The Partial Differential Equations

We wish to construct 2-ruled calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$ by solving partial differential equations for maps ϕ_1, ϕ_2, ψ. By Proposition 2.11, it is sufficient to consider the Cayley case.

Let Σ be a 2-dimensional, connected, real analytic manifold, let $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ be orthogonal real analytic maps such that $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^7$ defined by $\iota(\sigma, e^{i\theta}) = \cos\theta\phi_1(\sigma) + \sin\theta\phi_2(\sigma)$ is an immersion and let $\psi : \Sigma \to \mathbb{R}^8$ be a real analytic map. Clearly, $\mathbb{R}^2 \times \Sigma$ is an r-framed 2-ruled 4-fold with 2-ruling (Σ, π), where $\pi(r_1, r_2, \sigma) = \sigma$. Let M be defined by (8). Then M is the image of the map $\iota_M : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^8$ given by $\iota_M(r_1, r_2, \sigma) = r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma)$. Since ι is an immersion, ι_M is an immersion almost everywhere. Thus M is an r-framed 2-ruled 4-fold in $\mathbb{R}^8$, possibly with singularities.

Suppose that M is Cayley and let $p \in M$. Then there exist $(r_1, r_2) \in \mathbb{R}^2$, $\sigma \in \Sigma$ such that $p = r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma)$. Choose coordinates (s, t) near σ in Σ. Then $T_pM = \langle x, y, z, w\rangle_{\mathbb{R}}$, where $x = \phi_1(\sigma)$, $y = \phi_2(\sigma)$, $z = r_1\frac{\partial\phi_1}{\partial s}(\sigma) + r_2\frac{\partial\phi_2}{\partial s}(\sigma) + \frac{\partial\psi}{\partial s}(\sigma)$, $w = r_1\frac{\partial\phi_1}{\partial t}(\sigma) + r_2\frac{\partial\phi_2}{\partial t}(\sigma) + \frac{\partial\psi}{\partial t}(\sigma)$. The tangent space T_pM is a Cayley 4-plane. By Proposition 2.8, this is true if and only if $\mathrm{Im}(x \times y \times z \times w) = 0$. This implies that a quadratic in r_1, r_2 must vanish, but since this condition is forced to hold for all $(r_1, r_2) \in \mathbb{R}^2$, each coefficient in the quadratic is zero. Therefore the following set of equations must hold in Σ:

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0, \tag{11}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) = 0, \tag{12}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) + \mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0, \tag{13}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\psi}{\partial t}\right) = 0, \tag{14}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\psi}{\partial t}\right) + \mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0, \tag{15}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\psi}{\partial t}\right) + \mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) = 0. \tag{16}$$

If we do not suppose M to be Cayley but instead insist that (11)-(16) hold in Σ then, following the argument above, each tangent space to M must be Cayley and hence M is a Cayley 4-fold. Noting that (11)-(13) are precisely the conditions for the asymptotic cone M_0 of M to be Cayley, we deduce the following result.

Proposition 4.1 *Let M be a 2-ruled Cayley 4-fold in $\mathbb{R}^8$ and let M_0 be its asymptotic cone. Then M_0 is a 2-ruled Cayley cone in $\mathbb{R}^8$ provided that it is 4-dimensional.*

Clearly, M_0 is the image of the map $\iota_0 : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^8$ given by $\iota_0(r_1, r_2, \sigma) = r_1\phi_1(\sigma) + r_2\phi_2(\sigma)$. Since we suppose that ι is an immersion, ι_0 is an immersion except at $(r_1, r_2) = (0, 0)$, so M_0 is nonsingular except at 0 and thus is a cone.

Note that Φ is a nowhere vanishing 4-form on M_0 that defines its orientation, since M_0 is Cayley. Hence, if (s, t) are local coordinates on Σ, we can define them to be oriented by imposing

$$\Phi\left(\phi_1,\ \phi_2,\ r_1\frac{\partial\phi_1}{\partial s} + r_2\frac{\partial\phi_2}{\partial s},\ r_1\frac{\partial\phi_1}{\partial t} + r_2\frac{\partial\phi_2}{\partial t}\right) > 0 \tag{17}$$

for all $(r_1, r_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. It follows that

$$\Phi\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t}\right) > 0 \quad \text{and} \quad \Phi\left(\phi_1, \phi_2, \frac{\partial\phi_2}{\partial s}, \frac{\partial\phi_2}{\partial t}\right) > 0. \tag{18}$$

Consequently, $\{\phi_1, \phi_2, \frac{\partial\phi_j}{\partial s}, \frac{\partial\phi_j}{\partial t}\}$ is a linearly independent set for $j = 1, 2$. Moreover, (17) is equivalent to the condition that ι is an immersion.

We now construct a metric on Σ, under suitable conditions, using ϕ_1, ϕ_2 and the metric on $\mathbb{R}^8$. This enables us to formulate (11)-(16) as partial differential equations involving the triple cross product in $\mathbb{R}^8$, which we may write as

$$(x \times y \times z)^e = \Phi_{abcd}x^a y^b z^c g^{de} \tag{19}$$

using index notation for tensors on $\mathbb{R}^8$, where g^{de} is the inverse of the Euclidean metric on $\mathbb{R}^8$ and $x, y, z \in \mathbb{R}^8$. It can be easily verified, using (3) and a multiplication table for the octonions as given in A.1, that this definition coincides with (5). We immediately deduce that

$$\Phi(x, y, z, w) = g(x \times y \times z, w). \tag{20}$$

Note that the triple cross product $x \times y \times z$ is orthogonal to x, y, z, and that it is nonzero if and only if $\{x, y, z\}$ is a linearly independent set.

For a function $f : \Sigma \to \mathbb{R}^8$, we define $f^\perp : \Sigma \to \mathbb{R}^8$ by choosing $f^\perp(\sigma)$ to be the component of $f(\sigma)$ that lies in the orthogonal complement of $\langle\phi_1(\sigma), \phi_2(\sigma)\rangle_{\mathbb{R}}$. Since the fourfold cross product is alternating, (11)-(13) hold if and only if

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \left(\cos\theta\frac{\partial\phi_1}{\partial s}^{\perp} + \sin\theta\frac{\partial\phi_2}{\partial s}^{\perp}\right) \times \left(\cos\theta\frac{\partial\phi_1}{\partial t}^{\perp} + \sin\theta\frac{\partial\phi_2}{\partial t}^{\perp}\right)\right) = 0, \tag{21}$$

for all $\theta \in \mathbb{R}$. Let $\sigma \in \Sigma$. From Proposition 2.8 and (17) we see that, for each $\theta \in \mathbb{R}$, the four terms in (21), evaluated at σ, form a basis for a Cayley 4-plane Π_θ. By the definition of the triple cross product, we may also take $(\phi_1(\sigma), \phi_2(\sigma), \cos\theta \frac{\partial \phi_1}{\partial s}^\perp(\sigma) + \sin\theta \frac{\partial \phi_2}{\partial s}^\perp(\sigma), \cos\theta \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}^\perp(\sigma) + \sin\theta \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s}^\perp(\sigma))$ as a basis for Π_θ. Therefore,

$$\cos\theta \frac{\partial \phi_1}{\partial t}^\perp(\sigma) + \sin\theta \frac{\partial \phi_2}{\partial t}^\perp(\sigma) = A_\theta \left(\cos\theta \frac{\partial \phi_1}{\partial s}^\perp(\sigma) + \sin\theta \frac{\partial \phi_2}{\partial s}^\perp(\sigma) \right)$$
$$+ B_\theta \left(\cos\theta \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}^\perp(\sigma) + \sin\theta \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s}^\perp(\sigma) \right) \qquad (22)$$

for constants A_θ, B_θ depending on θ. We set $\theta = 0, \frac{\pi}{2}$ in (22) and substitute back in the expressions found for the t derivatives to obtain:

$$\cos\theta \left((A_0 - A_\theta) \frac{\partial \phi_1}{\partial s}^\perp(\sigma) + (B_0 - B_\theta) \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}^\perp(\sigma) \right) =$$
$$\sin\theta \left((A_\theta - A_{\frac{\pi}{2}}) \frac{\partial \phi_2}{\partial s}^\perp(\sigma) + (B_\theta - B_{\frac{\pi}{2}}) \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s}^\perp(\sigma) \right) \qquad (23)$$

To proceed in defining a metric on Σ we impose a condition on the dimension of

$$V_\sigma = \left\langle \frac{\partial \phi_1}{\partial s}^\perp(\sigma), \frac{\partial \phi_2}{\partial s}^\perp(\sigma), \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}^\perp(\sigma), \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s}^\perp(\sigma) \right\rangle_{\mathbb{R}}.$$

Let $W_\sigma = \langle \phi_1(\sigma), \phi_2(\sigma) \rangle_{\mathbb{R}}^\perp \subseteq \mathbb{R}^8$ and define $J : W_\sigma \to W_\sigma$ by $J(v) = \phi_1(\sigma) \times \phi_2(\sigma) \times v$. It is clear, through calculation in coordinates, that $J^2 = -1$ on W_σ. Note that $V_\sigma \subseteq W_\sigma$ is closed under the action of J, which can thus be considered as a form of complex structure on V_σ. Hence, V_σ is even dimensional. Since the case $\dim V_\sigma = 0$ is excluded by (18), $\dim V_\sigma = 2$ or 4. Recall that Σ is real analytic and connected. Therefore $\{\sigma \in \Sigma : \dim V_\sigma = 2\}$ is a closed real analytic subset of Σ and consequently either coincides with Σ or is of zero measure in Σ.

Suppose that $\dim V_\sigma = 4$. The four vectors in (23) are then linearly independent and hence

$$(A_0 - A_\theta)\cos\theta = (B_0 - B_\theta)\cos\theta = (A_{\frac{\pi}{2}} - A_\theta)\sin\theta = (B_{\frac{\pi}{2}} - B_\theta)\sin\theta = 0$$

for all θ. This clearly forces A_θ, B_θ to be constant. Let $A_\theta = A$ and $B_\theta = B$ for all θ, where A, B are real constants. We define a metric g_Σ on Σ pointwise by

the following equations:

$$g_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right) = A g_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right),\ g_\Sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = (A^2+B^2) g_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right). \quad (24)$$

Using (22) and the fact that $J^2 = -1$ on V_σ,

$$\begin{pmatrix} \phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial s}^\perp(\sigma) \\ \phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial t}^\perp(\sigma) \end{pmatrix} = K \begin{pmatrix} \frac{\partial \phi_j}{\partial s}^\perp(\sigma) \\ \frac{\partial \phi_j}{\partial t}^\perp(\sigma) \end{pmatrix},$$

for $j = 1, 2$, where K is a 2×2 matrix given by:

$$K = \frac{1}{B}\begin{pmatrix} -A & 1 \\ -(A^2+B^2) & A \end{pmatrix}.$$

If we change coordinates (s, t) to $(\tilde{s}, \tilde{t})$, with Jacobian matrix L, then K transforms to a matrix $\tilde{K} = LKL^{-1}$. We may then calculate the corresponding $\tilde{A}, \tilde{B}$ defining $\tilde{K}$ and see that they satisfy (24) for the coordinates $(\tilde{s}, \tilde{t})$. Therefore, g_Σ is a well-defined metric, up to scale, covariant under transformation of coordinates.

Having defined the metric g_Σ we can consider Σ as a *Riemannian* 2-fold, which has a natural orientation derived from the orientation on M and on the 2-planes $\langle \phi_1(\sigma), \phi_2(\sigma)\rangle_{\mathbb{R}}$. Therefore it has a natural *complex structure* which we denote as J. If we choose a local holomorphic coordinate $u = s + it$ on Σ, then the corresponding real coordinates must satisfy $\frac{\partial}{\partial t} = J\frac{\partial}{\partial s}$. We say that local real coordinates (s, t) on Σ satisfying this condition are *oriented conformal coordinates*. This forces $A = 0$, $B = 1$ in the notation of (24), since $B > 0$ by (18).

We now state and prove a theorem in this case.

Theorem 4.2 *Let Σ be a connected real analytic 2-fold, let $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ be orthogonal real analytic maps such that the map $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^7$ defined by $\iota(\sigma, e^{i\theta}) = \cos\theta\phi_1(\sigma) + \sin\theta\phi_2(\sigma)$ is an immersion, and let $\psi : \Sigma \to \mathbb{R}^8$ be a real analytic map. Define M by (8) and suppose that $\dim V_\sigma = 4$ almost everywhere in Σ. Then M is Cayley if and only if*

$$\frac{\partial \phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s} + f\phi_2, \quad (25)$$

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} - f\phi_1, \quad (26)$$

for some function $f : \Sigma \to \mathbb{R}$, and ψ satisfies

$$\frac{\partial \psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \psi}{\partial s} + g_1\phi_1 + g_2\phi_2 \quad (27)$$

for some functions $g_1, g_2 : \Sigma \to \mathbb{R}$, where the triple cross product is defined in (19) and (s,t) are oriented conformal coordinates on Σ. Moreover, sufficiency holds irrespective of the dimension of V_σ.

Proof: Recalling that (11)-(16) correspond to the condition that M is Cayley, we show that (11)-(13) are equivalent to (25)-(26), and that (14)-(16) are equivalent to (27).

Let $\sigma \in \Sigma$. Since ϕ_1 maps to $\mathcal{S}^7$ it is clear that $\phi_1(\sigma)$ is orthogonal to $\frac{\partial \phi_1}{\partial s}(\sigma)$ and $\frac{\partial \phi_1}{\partial t}(\sigma)$. By (22) and the work above, there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$\frac{\partial \phi_1}{\partial t}(\sigma) = a_1 \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}(\sigma) + a_2 \phi_2(\sigma) + a_3 \frac{\partial \phi_1}{\partial s}(\sigma). \tag{28}$$

We then calculate:

$$g\left(\frac{\partial \phi_1}{\partial t}^{\perp}(\sigma), \frac{\partial \phi_1}{\partial s}^{\perp}(\sigma)\right) = a_3 \left|\frac{\partial \phi_1}{\partial s}^{\perp}(\sigma)\right|^2.$$

The left-hand side is zero by (24) since (s,t) are oriented conformal coordinates, and hence $a_3 = 0$. We also have that

$$\left|\frac{\partial \phi_1}{\partial t}^{\perp}(\sigma)\right|^2 = a_1^2 \left|\phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}(\sigma)\right|^2 = a_1^2 \left|\frac{\partial \phi_1}{\partial s}^{\perp}(\sigma)\right|^2.$$

Therefore $a_1^2 = 1$ by (24). Further, taking the inner product of (28) with the triple cross product gives:

$$\begin{aligned}\left|\phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}(\sigma)\right|^2 a_1 &= g\left(\frac{\partial \phi_1}{\partial t}(\sigma), \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}(\sigma)\right) \\ &= \Phi\left(\phi_1(\sigma), \phi_2(\sigma), \frac{\partial \phi_1}{\partial s}(\sigma), \frac{\partial \phi_1}{\partial t}(\sigma)\right),\end{aligned}$$

using (19). Therefore $a_1 > 0$ by (18). Hence, $a_1 = 1$ and (25) holds at σ with $f(\sigma) = a_2$.

If (25) holds at σ then, by the definition of the triple cross product, the 4-plane spanned by $\{\phi_1(\sigma), \phi_2(\sigma), \frac{\partial \phi_1}{\partial s}(\sigma), \frac{\partial \phi_1}{\partial t}(\sigma)\}$ is Cayley.

Similarly, we deduce that (12) holding at σ is equivalent to

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} + f' \phi_1 \tag{29}$$

at σ, for some function $f' : \Sigma \to \mathbb{R}$. However,

$$\frac{\partial}{\partial t} g(\phi_1, \phi_2) = g\left(\frac{\partial \phi_1}{\partial t}, \phi_2\right) + g\left(\phi_1, \frac{\partial \phi_2}{\partial t}\right) = 0$$

and hence $f' = -f$.

It follows from [3, Theorem IV.1.38] that Spin(7) acts transitively upon oriented orthonormal bases of Cayley 4-planes. Therefore, we are always able to transform coordinates on $\mathbb{R}^8$ using Spin(7) such that a Cayley 4-plane has basis (e_1, e_2, e_3, e_4). In particular, any orthonormal pair can be mapped to the pair (e_1, e_2).

By the remarks above, we may transform coordinates on $\mathbb{R}^8$ using Spin(7) such that $\phi_1(\sigma) = e_1$, $\phi_2(\sigma) = e_2$, $\frac{\partial \phi_1}{\partial s}(\sigma) = b_1 e_1 + \ldots + b_8 e_8$, $\frac{\partial \phi_2}{\partial s}(\sigma) = b'_1 e_1 + \ldots + b'_8 e_8$ for some real constants $b_1, \ldots, b_8, b'_1, \ldots, b'_8$. If (25) and (26) hold, we may calculate $\frac{\partial \phi_1}{\partial t}(\sigma)$ and $\frac{\partial \phi_2}{\partial t}(\sigma)$. A straightforward calculation in coordinates then shows that (11)-(13) hold at σ. Since the triple cross product is invariant under Spin(7) by equation (19), we conclude that (11)-(13) are equivalent to (25) and (26).

Suppose now that (14)-(16) hold at $\sigma \in \Sigma$. Using Spin(7), transform coordinates such that $\phi_1(\sigma) = e_1$, $\phi_2(\sigma) = e_2$, $\frac{\partial \phi_1}{\partial s}(\sigma) = b_1 e_1 + \ldots + b_4 e_4$, where $b_1, \ldots, b_4$ are real constants, which we are free to do by equation (11). In these coordinates write $\frac{\partial \psi}{\partial s}(\sigma) = c_1 e_1 + \ldots + c_8 e_8$ and $\frac{\partial \psi}{\partial t}(\sigma) = d_1 e_1 + \ldots + d_8 e_8$. Calculating $\frac{\partial \phi_1}{\partial t}(\sigma)$ using (25), we then evaluate the terms in (15) as follows:

$$\begin{pmatrix} -b_4 & -b_3 \\ -b_3 & b_4 \end{pmatrix} \begin{pmatrix} d_5 + c_6 \\ d_6 - c_5 \end{pmatrix} = 0, \tag{30}$$

$$\begin{pmatrix} b_4 & b_3 \\ b_3 & -b_4 \end{pmatrix} \begin{pmatrix} d_7 + c_8 \\ d_8 - c_7 \end{pmatrix} = 0. \tag{31}$$

Details of the calculation of the fourfold cross product may be found in A.2. The determinant of the matrices in (30) and (31) is $-b_3^2 - b_4^2 \neq 0$, since $\frac{\partial \phi_1}{\partial s}(\sigma) \notin \langle \phi_1(\sigma), \phi_2(\sigma) \rangle_{\mathbb{R}}$. Therefore $d_5 = -c_6$, $d_6 = c_5$, $d_7 = -c_8$ and $d_8 = c_7$.

We may also evaluate (14):

$$c_5 d_8 + c_6 d_7 - c_7 d_6 - c_8 d_5 = 0, \tag{32}$$

$$c_5 d_7 - c_6 d_8 - c_7 d_5 + c_8 d_6 = 0, \tag{33}$$

$$-c_3 d_8 + c_7 d_4 - c_4 d_7 + c_8 d_3 = 0, \tag{34}$$

$$-c_3 d_7 + c_4 d_8 - c_8 d_4 + c_7 d_3 = 0, \tag{35}$$

$$c_3 d_6 + c_4 d_5 - c_5 d_4 - c_6 d_3 = 0, \tag{36}$$

$$c_3 d_5 - c_4 d_6 - c_5 d_3 + c_6 d_4 = 0. \tag{37}$$

Again, details may be found in A.2. Substituting in the results above, we have

that (32)-(33) are satisfied trivially and (34)-(37) become:

$$\begin{pmatrix} c_8 & c_7 \\ c_7 & -c_8 \end{pmatrix} \begin{pmatrix} d_3 + c_4 \\ d_4 - c_3 \end{pmatrix} = 0, \tag{38}$$

$$\begin{pmatrix} -c_6 & -c_5 \\ -c_5 & c_6 \end{pmatrix} \begin{pmatrix} d_3 + c_4 \\ d_4 - c_3 \end{pmatrix} = 0. \tag{39}$$

We deduce that the determinants of the matrices in (38) and (39) are zero, or the vector appearing in both equations is zero. Therefore,

(i) $d_3 = -c_4$ and $d_4 = c_3$

or

(ii) $c_5 = c_6 = c_7 = c_8 = 0$.

Condition (i) implies that (27) holds at σ with $g_1(\sigma) = d_1$, $g_2(\sigma) = d_2$, by the definition of the triple cross product and its invariance under Spin(7). Condition (ii) corresponds to

$$\frac{\partial \psi}{\partial s}(\sigma), \frac{\partial \psi}{\partial t}(\sigma) \in \left\langle \phi_1(\sigma), \phi_2(\sigma), \frac{\partial \phi_j}{\partial s}(\sigma), \frac{\partial \phi_j}{\partial t}(\sigma) \right\rangle_{\mathbb{R}} \tag{40}$$

holding for $j = 1$. Therefore, (14) and (15) are equivalent to (27) or (40) for $j = 1$ holding at σ. We similarly deduce that (14) and (16) are equivalent to (27) or (40) for $j = 2$ holding at σ.

We conclude that (11)-(16) are equivalent to (25), (26) and condition (27) or (40) for $j = 1, 2$ at each point $\sigma \in \Sigma$. Recall that Σ is connected and ϕ_1, ϕ_2, ψ and Σ are real analytic. Note that $\Sigma_1 = \{\sigma \in \Sigma : \dim V_\sigma = 4\}$ is an open subset of Σ whose complement is measure zero in Σ by hypothesis.

Let $\sigma \in \Sigma_1$ and suppose that (40) for $j = 1, 2$ holds at σ. Then, there exist real constants C_{jk}, for $j = 1, 2$, $1 \leq k \leq 4$, such that

$$\frac{\partial \psi}{\partial s}(\sigma) = C_{j1}\phi_1(\sigma) + C_{j2}\phi_2(\sigma) + C_{j3}\frac{\partial \phi_j}{\partial s}^{\perp}(\sigma) + C_{j4}\frac{\partial \phi_j}{\partial t}^{\perp}(\sigma).$$

Clearly, $C_{1k} = C_{2k}$ for $k = 1, 2$ by the definition of $g^\perp$ for a function g. Moreover, since $\dim V_\sigma = 4$ ensures the linear independence of the partial derivatives of ϕ_1 and ϕ_2, $C_{jk} = 0$ for $j = 1, 2$, $k = 3, 4$. Hence, $\frac{\partial \psi}{\partial s}(\sigma)$ and, similarly, $\frac{\partial \psi}{\partial t}(\sigma)$ lie in $\langle \phi_1(\sigma), \phi_2(\sigma) \rangle_{\mathbb{R}}$ for almost all $\sigma \in \Sigma$. Therefore ψ satisfies (27).

Consequently, (27) holds in Σ_1. Moreover, $\Sigma_2 = \{\sigma \in \Sigma : (27) \text{ holds at } \sigma\}$ is a closed real analytic subset of Σ and so must either coincide with Σ or be

of zero measure in Σ. Since $\Sigma_1 \subseteq \Sigma_2$, Σ_2 cannot be measure zero and so must equal Σ. This completes the proof. $\square$

Note that (27) is a *linear* condition on ψ given ϕ_1 and ϕ_2, and that (25) and (26) are equivalent to the fact that the asymptotic cone M_0 of M is Cayley. Therefore, if we are given an r-framed, 2-ruled Cayley cone M_0 defined by ϕ_1 and ϕ_2, then any solution ψ of (27), together with ϕ_1 and ϕ_2, defines an r-framed 2-ruled Cayley 4-fold with asymptotic cone M_0. We also note that (27) is unchanged if ϕ_1 and ϕ_2 are fixed and satisfy (25) and (26), but ψ is replaced by $\psi + \tilde{g}_1\phi_1 + \tilde{g}_2\phi_2$ for real analytic maps $\tilde{g}_1, \tilde{g}_2$. We can thus locally transform ψ such that g_1 and g_2 are zero.

If we suppose instead that $\dim V_\sigma = 2$ for all $\sigma \in \Sigma$ then we are unable, in general, to define a suitable metric and hence oriented conformal coordinates on Σ. However, we shall show that if we exclude *planar* r-framed 2-ruled 4-folds, then (25)-(27) of Theorem 4.2 characterize the Cayley condition on ϕ_1, ϕ_2, ψ and that there is a natural conformal structure on Σ.

4.1 Gauge Transformations

Let ϕ_1, ϕ_2 satisfy (25) and (26) in Theorem 4.2 for some map f. Taking the triple cross product of (25) and (26) with ϕ_1 and ϕ_2 gives:

$$\frac{\partial \phi_1}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial t} + f^{'}\phi_2, \tag{41}$$

$$\frac{\partial \phi_2}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial t} - f^{'}\phi_1, \tag{42}$$

for some function $f^{'} : \Sigma \to \mathbb{R}$.

We are allowed to perform a rotation $\Theta(\sigma)$ to the $(\phi_1(\sigma), \phi_2(\sigma))$-plane at each point $\sigma \in \Sigma$ as long as the function Θ is smooth. The choice of Θ will then alter f and $f^{'}$. We call such a transformation a *gauge transformation*.

We now show that under certain conditions there exists a gauge transformation such that $f = f^{'} = 0$. Let $\Theta : \Sigma \to \mathbb{R}$ be a smooth function and define $\tilde{\phi}_1, \tilde{\phi}_2$ by

$$\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Then $\tilde{\phi}_1, \tilde{\phi}_2$ satisfy (25) and (26) with f replaced by $\tilde{f} = f + \frac{\partial\Theta}{\partial t}$. Moreover, they satisfy (41) and (42) with f' replaced by $\tilde{f}' = f' + \frac{\partial\Theta}{\partial s}$. Therefore, locally, there exists a smooth function Θ such that $\tilde{f} = \tilde{f}' = 0$ if and only if $\frac{\partial f}{\partial s} = \frac{\partial f'}{\partial t}$.

If we differentiate (25) with respect to s and differentiate (41) with respect to t we get:

$$\frac{\partial^2\phi_1}{\partial s\partial t} = \phi_1 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2\phi_1}{\partial s^2} + \frac{\partial f}{\partial s}\,\phi_2 + f\,\frac{\partial\phi_2}{\partial s}, \tag{43}$$

$$\frac{\partial^2\phi_1}{\partial t\partial s} = -\phi_1 \times \frac{\partial\phi_2}{\partial t} \times \frac{\partial\phi_1}{\partial t} - \phi_1 \times \phi_2 \times \frac{\partial^2\phi_1}{\partial t^2} + \frac{\partial f'}{\partial t}\phi_2 + f'\frac{\partial\phi_2}{\partial t}. \tag{44}$$

We must have that (43) and (44) are equal. In particular, the inner products of ϕ_2 with (43) and (44) must be equal. Note that

$$g\left(\phi_2, \frac{\partial^2\phi_1}{\partial s\partial t}\right) = -\Phi\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}\right) + \frac{\partial f}{\partial s},$$

$$g\left(\phi_2, \frac{\partial^2\phi_1}{\partial t\partial s}\right) = \Phi\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}\right) + \frac{\partial f'}{\partial t}$$

and that

$$\Phi\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}\right) = g\left(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}\right), \tag{45}$$

$$\Phi\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}\right) = -g\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial t}\right). \tag{46}$$

However,

$$\Phi\left(\phi_1, \phi_2, \frac{\partial\phi_2}{\partial s}, \frac{\partial\phi_1}{\partial s}\right) = g\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial t}\right).$$

Hence, since Φ is alternating, we deduce that $\frac{\partial f}{\partial s} = \frac{\partial f'}{\partial t}$ if and only if all the terms in (45) and (46) are zero.

We say that functions ϕ_1 and ϕ_2 satisfying (25), (26), (41) and (42) with $f = f' = 0$ are in the *flat gauge*.

We now give a geometric interpretation of the flat gauge. Let (Σ, π) be a 2-ruling. Then, there is an $\mathcal{S}^1$ bundle $\pi_P : P \to \Sigma$ as described after Definition 3.1. An r-framing, which is equivalent to a choice of ϕ_1, ϕ_2, gives a trivialization of P and we can consider it as a U(1) bundle. Define a connection ∇_P on P by a connection 1-form given by $d\theta - f'ds - fdt$, where θ corresponds to the U(1) direction. This connection is independent of the choice of r-framing, by the work above, and has curvature which may be written as $(\frac{\partial f'}{\partial t} - \frac{\partial f}{\partial s})ds \wedge dt$. Hence, the connection ∇_P defined by ϕ_1, ϕ_2 is flat if and only if ϕ_1, ϕ_2 can be put in the flat gauge by some gauge transformation locally.

4.2 Planar 2-ruled Cayley 4-folds

In this subsection we show that maps ϕ_1, ϕ_2, ψ which do not satisfy (25)-(27) for any local oriented coordinates (s, t) on Σ define a *planar* Cayley 4-fold.

The next result shows that (25)-(27) can be considered as *evolution equations* for ϕ_1, ϕ_2, ψ. We make the definition here that a function is real analytic on a compact interval I in $\mathbb{R}$ if it extends to a real analytic function on an open set containing I.

Theorem 4.3 *Let I be a compact interval in $\mathbb{R}$, let s be a coordinate on I, let $\phi_1', \phi_2' : I \to \mathcal{S}^7$ be orthogonal real analytic maps and let $\psi' : I \to \mathbb{R}^8$ be a real analytic map. Let N be a neighbourhood of 0 in $\mathbb{R}$ and let $f : I \times N \to \mathbb{R}$ be a real analytic map. Then there exist $\epsilon > 0$ and unique real analytic maps $\phi_1, \phi_2 : I \times (-\epsilon, \epsilon) \to \mathcal{S}^7$, with ϕ_1, ϕ_2 orthogonal, and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^8$ satisfying $\phi_1(s, 0) = \phi_1'(s)$, $\phi_2(s, 0) = \phi_2'(s)$, $\psi(s, 0) = \psi'(s)$ for all $s \in I$ and*

$$\frac{\partial \phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s} + f\phi_2, \tag{47}$$

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} - f\phi_1, \tag{48}$$

$$\frac{\partial \psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \psi}{\partial s}, \tag{49}$$

where t is a coordinate on $(-\epsilon, \epsilon)$ and the triple cross product is defined in (19). *Let M be defined by:*

$$M = \{r_1\phi_1(s,t) + r_2\phi_2(s,t) + \psi(s,t) : (r_1, r_2) \in \mathbb{R}^2,\ s \in I,\ t \in (-\epsilon, \epsilon)\}. \tag{50}$$

Then M is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$.

Proof: Since I is compact and $\phi_1', \phi_2', \psi', f$ are real analytic, we may apply the *Cauchy–Kowalevsky Theorem* [7, p. 234] from the theory of partial differential equations to give unique functions $\phi_1, \phi_2, \psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^8$ satisfying the initial conditions and (47)-(49). We must now show that ϕ_1, ϕ_2 map to $\mathcal{S}^7$ and are orthogonal.

We first note that

$$\begin{aligned}
\frac{\partial}{\partial t}\, g(\phi_1, \phi_2) &= g\left(\frac{\partial \phi_1}{\partial t}, \phi_2\right) + g\left(\phi_1, \frac{\partial \phi_2}{\partial t}\right) \\
&= f\left(g(\phi_2, \phi_2) - g(\phi_1, \phi_1)\right), \\
\frac{\partial}{\partial t}\, g(\phi_1, \phi_1) &= 2fg(\phi_1, \phi_2), \\
\frac{\partial}{\partial t}\, g(\phi_2, \phi_2) &= -2fg(\phi_1, \phi_2).
\end{aligned}$$

Then $g(\phi_j, \phi_k)$ for $j, k = 1, 2$ are real analytic functions satisfying this system of partial differential equations, together with the initial conditions $g(\phi_1, \phi_1) = g(\phi_2, \phi_2) = 1$ and $g(\phi_1, \phi_2) = \frac{\partial}{\partial t} g(\phi_j, \phi_k) = 0$ at $t = 0$ given by assumption. The

functions $g(\phi_1, \phi_1) = g(\phi_2, \phi_2) \equiv 1$ and $g(\phi_1, \phi_2) \equiv 0$ also satisfy these equations and initial conditions. It therefore follows from the Cauchy–Kowalevsky Theorem that these two solutions must be locally equal and hence, for $\epsilon > 0$ sufficiently small, $|\phi_1| = |\phi_2| = 1$ and ϕ_1, ϕ_2 are orthogonal.

We conclude from Theorem 4.2 that M is an r-framed 2-ruled Cayley 4-fold. $\square$

Note that the Cayley 4-fold M resulting from Theorem 4.3 does not depend on the function f.

Let (Σ, π) and $(\tilde{\Sigma}, \tilde{\pi})$ be 2-rulings of a 4-fold in $\mathbb{R}^n$. We say that these 2-rulings are *distinct* if the families of affine 2-planes, $\mathcal{F}_\Sigma = \{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ and $\mathcal{F}_{\tilde{\Sigma}} = \{\tilde{\pi}^{-1}(\tilde{\sigma}) : \tilde{\sigma} \in \tilde{\Sigma}\}$, are different. If $\mathcal{F}^n$ is the family of all affine 2-planes in $\mathbb{R}^n$ we can consider (Σ, π) as a map from Σ to $\mathcal{F}^n$ given by $\sigma \mapsto \pi^{-1}(\sigma)$ with image $\mathcal{F}_\Sigma$.

Our next result is analogous to [2, Theorem 6 part 2], which relates to ruled SL 3-folds.

Proposition 4.4 *A 2-ruled Cayley 4-fold in $\mathbb{R}^8$ which admits a real analytic one-parameter family of distinct real analytic 2-rulings is locally isomorphic to an affine Cayley 4-plane in $\mathbb{R}^8$.*

Proof: Let $\{(\Sigma_u, \pi_u) : u \in \mathbb{R}\}$ be a real analytic family of distinct real analytic 2-rulings for a Cayley 4-fold M. Then there exists $p \in M$ such that $\Pi_u = \pi_u^{-1}(\pi_u(p))$ is not constant as a 2-plane in $\mathbb{R}^8$. Hence we have a one-parameter family of planes $\Pi_u \ni p$ in M such that $\frac{d\Pi_u}{du} \neq 0$ for some u, i.e. such that Π_u changes nontrivially. Therefore $\{\Pi_u : u \in \mathbb{R}\}$ is a real analytic one-dimensional family of planes in M containing p. The total space of this family is a real analytic 3-fold N contained in M. Moreover, every plane in M containing p lies in the affine Cayley 4-plane $p + T_pM$ and thus $N \subseteq p + T_pM$. By Theorem 2.10, M and $p + T_pM$ must coincide on a connected component of M. The result follows. $\square$

We now give the result claimed at the start of the subsection, which is analogous to the result [5, Proposition 5.3] for ruled SL 3-folds.

Proposition 4.5 *Any r-framed 2-ruled Cayley 4-fold (M, Σ, π) in $\mathbb{R}^8$ defined locally by maps ϕ_1, ϕ_2, ψ which do not satisfy (25)-(27) for any local oriented coordinates (s, t) on Σ is locally isomorphic to an affine Cayley 4-plane in $\mathbb{R}^8$.*

Proof: We may take the 2-ruling (Σ, π) to be locally real analytic since M is real analytic by Theorem 2.9. Let $I = [0, 1]$ and let $\gamma : I \to \Sigma$ be a real analytic

curve in Σ. If we set $\phi_1'(s) = \phi_1(\gamma(s))$, $\phi_2'(s) = \phi_2(\gamma(s))$, $\psi'(s) = \psi(\gamma(s))$, then by Theorem 4.3 we construct $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\psi}$ defining an r-framed 2-ruled Cayley 4-fold $\tilde{M}$ satisfying (25)-(27) of Theorem 4.2. We have that $M, \tilde{M}$ coincide in the real analytic 3-fold $\pi^{-1}(\gamma(I))$, and hence, by Theorem 2.10, they must be locally equal. Therefore, M locally admits a 2-ruling $(\tilde{\Sigma}, \tilde{\pi})$ satisfying (25)-(27) of Theorem 4.2, which must be distinct from (Σ, π).

Using the notation introduced before Proposition 4.4, the families of affine 2-planes $\mathcal{F}_\Sigma$ and $\mathcal{F}_{\tilde{\Sigma}}$ coincide in the family of affine 2-planes defined by points on γ, denoted $\mathcal{F}_\gamma$. Using local real analyticity of the families, either $\mathcal{F}_\Sigma$ is equal to $\mathcal{F}_{\tilde{\Sigma}}$ locally or they only meet in $\mathcal{F}_\gamma$ locally. The former possibility is excluded because the 2-rulings (Σ, π) and $(\tilde{\Sigma}, \tilde{\pi})$ are distinct and hence the latter is true.

Let γ_1 and γ_2 be distinct real analytic curves near γ in Σ defining 2-rulings (Σ_1, π_1) and (Σ_2, π_2), respectively, as above. Then $\mathcal{F}_\Sigma \cap \mathcal{F}_{\Sigma_j}$ is locally equal to $\mathcal{F}_{\gamma_j}$ for $j = 1, 2$. Hence, the 2-rulings (Σ_1, π_1) and (Σ_2, π_2) are not distinct (that is, $\mathcal{F}_{\Sigma_1} = \mathcal{F}_{\Sigma_2}$) if and only if $\mathcal{F}_{\gamma_1} = \mathcal{F}_{\gamma_2}$, which implies that $\gamma_1 = \gamma_2$. Therefore, distinct curves near γ in Σ produce different 2-rulings of M and hence M has infinitely many 2-rulings.

Let $\{\gamma_u : u \in \mathbb{R}\}$ be a one parameter family of distinct real analytic curves near γ in Σ with $\gamma_0 = \gamma$. Each curve in the family defines a distinct real analytic 2-ruling (Σ_u, π_u). Applying Proposition 4.4 gives the result. $\square$

Note that in the proof of Theorem 4.2 the condition (27) on ψ was forced by the linear independence of the derivatives of ϕ_1, ϕ_2. However, as we shall see in §5.2, non-planar 2-ruled 4-folds can be constructed when the derivatives of ϕ_1, ϕ_2 are linearly dependent.

Proposition 4.5 tells us that for any *non-planar* 2-ruled Cayley 4-fold M defined by maps ϕ_1, ϕ_2, ψ on Σ there exist locally oriented coordinates (s, t) on Σ such that (25)-(27) are satisfied. We shall see in the next subsection that there is therefore a natural conformal structure upon Σ, and (s, t) are oriented conformal coordinates with respect to this structure.

4.3 Main Results

We now present the main results of the paper on 2-ruled calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$. The first follows immediately from Theorem 4.2 and Proposition 4.5.

Theorem 4.6 *Let (M, Σ, π) be a non-planar, r-framed, 2-ruled 4-fold in $\mathbb{R}^8$ defined by orthogonal real analytic maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ and a real analytic*

map $\psi : \Sigma \to \mathbb{R}^8$ as follows:

$$M = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma) : r_1, r_2 \in \mathbb{R},\ \sigma \in \Sigma\}. \tag{51}$$

Then M is Cayley if and only if there exist locally oriented coordinates (s,t) on Σ such that

$$\frac{\partial\phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} + f\phi_2, \tag{52}$$

$$\frac{\partial\phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} - f\phi_1, \tag{53}$$

$$\frac{\partial\psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} + g_1\phi_1 + g_2\phi_2, \tag{54}$$

where the triple cross product is defined by equation (19) *and $f, g_1, g_2 : \Sigma \to \mathbb{R}$ are some real analytic functions.*

We now prove the result claimed at the end of the last subsection.

Proposition 4.7 *Let (M, Σ, π) be a non-planar, r-framed, 2-ruled Cayley 4-fold in $\mathbb{R}^8$. Then there exists a unique conformal structure on Σ with respect to which (s,t) as given in Theorem 4.6 are oriented conformal coordinates.*

Proof: Let (s,t) be local oriented coordinates as given by Theorem 4.6. Define a complex structure J on Σ by requiring that $u = s + it$ is a holomorphic coordinate on Σ, i.e. that $\frac{\partial}{\partial t} = J\frac{\partial}{\partial s}$. Note that ϕ_1, ϕ_2 as given in Theorem 4.6 satisfy

$$\frac{\partial\phi_j}{\partial t}^{\perp} = \phi_1 \times \phi_2 \times \frac{\partial\phi_j}{\partial s}^{\perp}, \tag{55}$$

$$\frac{\partial\phi_j}{\partial s}^{\perp} = -\phi_1 \times \phi_2 \times \frac{\partial\phi_j}{\partial t}^{\perp}, \tag{56}$$

for $j = 1, 2$. Suppose that $(\tilde{s}, \tilde{t})$ are local oriented coordinates on Σ such that ϕ_1, ϕ_2 also satisfy (52)-(53) in these coordinates. Hence, ϕ_1, ϕ_2 satisfy (55)-(56) for the coordinates $(\tilde{s}, \tilde{t})$.

We then calculate:

$$\begin{aligned}\frac{\partial\phi_j}{\partial\tilde{t}}^{\perp} &= \frac{\partial s}{\partial\tilde{t}}\frac{\partial\phi_j}{\partial s}^{\perp} + \frac{\partial t}{\partial\tilde{t}}\frac{\partial\phi_j}{\partial t}^{\perp} \\ &= \phi_1 \times \phi_2 \times \left(\frac{\partial t}{\partial\tilde{t}}\frac{\partial\phi_j}{\partial s}^{\perp} - \frac{\partial s}{\partial\tilde{t}}\frac{\partial\phi_j}{\partial t}^{\perp}\right)\end{aligned}$$

and

$$\frac{\partial\phi_j}{\partial\tilde{s}}^{\perp} = \frac{\partial s}{\partial\tilde{s}}\frac{\partial\phi_j}{\partial s}^{\perp} + \frac{\partial t}{\partial\tilde{s}}\frac{\partial\phi_j}{\partial t}^{\perp}.$$

Note that, from (55), $\frac{\partial\phi_j}{\partial t}^{\perp}$ is orthogonal to $\frac{\partial\phi_j}{\partial s}^{\perp}$ and, moreover, that $\frac{\partial\phi_j}{\partial t}^{\perp}\neq 0$ if and only if $\frac{\partial\phi_j}{\partial s}^{\perp}\neq 0$ by the definition of $f^{\perp}$ for a function $f:\Sigma\to\mathbb{R}$ and the properties of the triple cross product. Using (55)-(56) for $(\tilde{s},\tilde{t})$ we deduce that

$$\frac{\partial s}{\partial\tilde{s}}=\frac{\partial t}{\partial\tilde{t}}\quad\text{and}\quad\frac{\partial s}{\partial\tilde{t}}=-\frac{\partial t}{\partial\tilde{s}}\,,\tag{57}$$

since not both $\frac{\partial\phi_1}{\partial s}^{\perp}, \frac{\partial\phi_2}{\partial s}^{\perp}$ are zero.

Therefore, using (57),

$$\frac{\partial}{\partial\tilde{t}}=\frac{\partial s}{\partial\tilde{t}}\frac{\partial}{\partial s}+\frac{\partial t}{\partial\tilde{t}}\frac{\partial}{\partial t}=-\frac{\partial t}{\partial\tilde{s}}\frac{\partial}{\partial s}+\frac{\partial s}{\partial\tilde{s}}\frac{\partial}{\partial t}=J\left(\frac{\partial t}{\partial\tilde{s}}\frac{\partial}{\partial t}+\frac{\partial s}{\partial\tilde{s}}\frac{\partial}{\partial s}\right)=J\frac{\partial}{\partial\tilde{s}}.$$

Hence we have the result. □

It is clear that the conformal structure given by Proposition 4.7 coincides with that given by the metric as described in the preamble to Theorem 4.2.

We now use Proposition 2.11 in order to prove analogous results for SL 4-folds in $\mathbb{C}^4$ and coassociative 4-folds in $\mathbb{R}^7$.

We begin with the SL case and define the triple cross product of x, y, z in $\mathbb{C}^4$ by

$$(x\times y\times z)^e=(\operatorname{Re}\Omega)_{abcd}x^ay^bz^cg^{de},\tag{58}$$

using index notation for tensors on $\mathbb{C}^4$, where g^{de} is the inverse of the Euclidean metric on $\mathbb{C}^4$. By equation (7), this triple cross product agrees with the one defined in (19) when $\omega(x,y)=\omega(y,z)=\omega(z,x)=0$.

Theorem 4.8 *Let (M,Σ,π) be a non-planar, r-framed, 2-ruled 4-fold in $\mathbb{C}^4\cong\mathbb{R}^8$ defined by orthogonal real analytic maps $\phi_1,\phi_2:\Sigma\to\mathcal{S}^7$ and a real analytic map $\psi:\Sigma\to\mathbb{R}^8$ as follows:*

$$M=\{r_1\phi_1(\sigma)+r_2\phi_2(\sigma)+\psi(\sigma):r_1,r_2\in\mathbb{R},\,\sigma\in\Sigma\}.\tag{59}$$

Then M is special Lagrangian if and only if $\omega(\phi_1,\phi_2)\equiv 0$ and there exist locally oriented coordinates (s,t) on Σ such that:

$$\omega\left(\phi_j,\frac{\partial\phi_k}{\partial s}\right)\equiv 0\qquad\textit{for } j,k=1,2,\tag{60}$$

$$\omega\left(\phi_j,\frac{\partial\psi}{\partial s}\right)\equiv 0\qquad\textit{for } j=1,2,\tag{61}$$

$$\frac{\partial\phi_1}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}+f\phi_2,\tag{62}$$

$$\frac{\partial\phi_2}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}-f\phi_1,\tag{63}$$

$$\frac{\partial\psi}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\psi}{\partial s}+g_1\phi_1+g_2\phi_2,\tag{64}$$

where the triple cross product is defined by equation (58)*, and* $f, g_1, g_2 : \Sigma \to \mathbb{R}$ *are some real analytic functions.*

It is worth making clear that (62)-(64) are not the same as (52)-(54) because of the different definitions of the triple cross product.

Proof: By Proposition 2.11, M is SL if and only if M is Cayley and $\omega|_M \equiv 0$. We thus conclude from Theorem 4.6 that M is SL if and only if ϕ_1, ϕ_2, ψ satisfy (52)-(54) and $\omega|_{T_pM} \equiv 0$ for all $p \in M$. Therefore ω vanishes on $\langle x, y, z, w\rangle_{\mathbb{R}}$, where $x = \phi_1(\sigma)$, $y = \phi_2(\sigma)$, $z = r_1 \frac{\partial \phi_1}{\partial s}(\sigma) + r_2 \frac{\partial \phi_2}{\partial s}(\sigma) + \frac{\partial \psi}{\partial s}(\sigma)$, $w = r_1 \frac{\partial \phi_1}{\partial t}(\sigma) + r_2 \frac{\partial \phi_2}{\partial t}(\sigma) + \frac{\partial \psi}{\partial t}(\sigma)$, for all $(r_1, r_2) \in \mathbb{R}^2$, $\sigma \in \Sigma$. Hence, the equations that must be satisfied are $\omega(\phi_1, \phi_2) \equiv 0$ and

$$\omega\left(\phi_j, \frac{\partial \phi_k}{\partial s}\right) = \omega\left(\phi_j, \frac{\partial \phi_k}{\partial t}\right) \equiv 0 \qquad \text{for } j, k = 1, 2, \tag{65}$$

$$\omega\left(\phi_j, \frac{\partial \psi}{\partial s}\right) = \omega\left(\phi_j, \frac{\partial \psi}{\partial t}\right) \equiv 0 \qquad \text{for } j = 1, 2, \tag{66}$$

$$\omega\left(\frac{\partial \phi_j}{\partial s}, \frac{\partial \phi_j}{\partial t}\right) = \omega\left(\frac{\partial \psi}{\partial s}, \frac{\partial \psi}{\partial t}\right) \equiv 0 \qquad \text{for } j = 1, 2, \tag{67}$$

$$\omega\left(\frac{\partial \phi_1}{\partial s}, \frac{\partial \phi_2}{\partial t}\right) + \omega\left(\frac{\partial \phi_2}{\partial s}, \frac{\partial \phi_1}{\partial t}\right) \equiv 0, \tag{68}$$

$$\omega\left(\frac{\partial \phi_j}{\partial s}, \frac{\partial \psi}{\partial t}\right) + \omega\left(\frac{\partial \psi}{\partial s}, \frac{\partial \phi_j}{\partial t}\right) \equiv 0 \qquad \text{for } j = 1, 2. \tag{69}$$

However, if the functions ϕ_1, ϕ_2, ψ satisfy (65)-(69) and (52)-(54), then they must satisfy (62)-(64). Hence, it is enough to show that the conditions in the theorem force (65)-(69) to hold in order to prove the result.

If x, y, z, w are vectors in $\mathbb{C}^4$ such that ω vanishes on $\langle x, y, z, w\rangle_{\mathbb{R}}$, then direct calculation in coordinates shows that

$$\omega(x, y \times z \times w) = \mathrm{Im}(\epsilon_{abcd} x^a y^b z^c w^d), \tag{70}$$

using index notation for tensors on $\mathbb{C}^4$, where ϵ_{abcd} is the permutation symbol and the triple cross product is given in (58). Noting that $\omega(\phi_1, \phi_2) \equiv 0$, that (60) and (61) hold, and the relationship between the triple cross products on $\mathbb{C}^4$ and $\mathbb{R}^8$, we see that (62)-(64) hold. Hence $\omega(\phi_j, \frac{\partial \phi_k}{\partial t}) = 0$ for all j, k using (62)-(63) and (70). Therefore (65) is satisfied. Moreover, (64) and (70) imply that (66) is satisfied. If we use (60)-(61), (62)-(64) and (70) again, we have that (67) is satisfied.

We now show that (68) and (69) are satisfied. Calculation using (60), (62)-(63) and (70) gives:

$$\begin{aligned}\omega\left(\frac{\partial\phi_1}{\partial s},\frac{\partial\phi_2}{\partial t}\right)+\omega\left(\frac{\partial\phi_2}{\partial s},\frac{\partial\phi_1}{\partial t}\right)&\\&=\omega\left(\frac{\partial\phi_1}{\partial s},\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}\right)+\omega\left(\frac{\partial\phi_2}{\partial s},\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}\right)\\&=\mathrm{Im}\left(\epsilon_{abcd}\frac{\partial\phi_1}{\partial s}^a\phi_1^b\phi_2^c\frac{\partial\phi_2}{\partial s}^d\right)+\mathrm{Im}\left(\epsilon_{abcd}\frac{\partial\phi_2}{\partial s}^a\phi_1^b\phi_2^c\frac{\partial\phi_1}{\partial s}^d\right)\\&=\mathrm{Im}\left((\epsilon_{abcd}+\epsilon_{dbca})\frac{\partial\phi_1}{\partial s}^a\phi_1^b\phi_2^c\frac{\partial\phi_2}{\partial s}^d\right)\equiv 0\end{aligned}$$

by the definition of the permutation symbol. Hence (68) is satisfied. An entirely similar argument using (60)-(64) and (70) gives that (69) is satisfied. □

For the coassociative case we define the triple cross product of x, y, z in $\mathbb{R}^7$ by

$$(x\times y\times z)^e=(*\varphi)_{abcd}x^a y^b z^c g^{de} \tag{71}$$

using index notation for tensors on $\mathbb{R}^7$, where g^{de} is the inverse of the Euclidean metric on $\mathbb{R}^7$. If we embed $\mathbb{R}^7$ as $\{0\}\times\mathbb{R}^7$ in $\mathbb{R}^8$, then (6) implies that this triple cross product agrees with (19) when $\varphi(x, y, z) = 0$.

Theorem 4.9 *Let (M, Σ, π) be a non-planar, r-framed, 2-ruled 4-fold in $\mathbb{R}^7$ defined by orthogonal real analytic maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^6$, and a real analytic map $\psi : \Sigma \to \mathbb{R}^7$ as follows:*

$$M=\{r_1\phi_1(\sigma)+r_2\phi_2(\sigma)+\psi(\sigma): r_1, r_2\in\mathbb{R},\ \sigma\in\Sigma\}. \tag{72}$$

Then M is coassociative if and only if there exist locally oriented coordinates (s, t) on Σ such that

$$\varphi\left(\phi_1,\phi_2,\frac{\partial\phi_j}{\partial s}\right)\equiv 0 \qquad \textit{for } j=1,2, \tag{73}$$

$$\varphi\left(\phi_1,\phi_2,\frac{\partial\psi}{\partial s}\right)\equiv 0, \tag{74}$$

$$\frac{\partial\phi_1}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}+f\phi_2, \tag{75}$$

$$\frac{\partial\phi_2}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}-f\phi_1, \tag{76}$$

$$\frac{\partial\psi}{\partial t}=\phi_1\times\phi_2\times\frac{\partial\psi}{\partial s}+g_1\phi_1+g_2\phi_2, \tag{77}$$

where the triple cross product is defined by equation (71), and $f, g_1, g_2 : \Sigma \to \mathbb{R}$ are some real analytic functions.

Proof: By Proposition 2.11, M is coassociative if and only if $M \subseteq \mathbb{R}^7 \subseteq \mathbb{R}^8$ is Cayley. We may deduce from Theorem 4.6 that M is Cayley if and only if there exist locally coordinates (s,t) such that ϕ_1, ϕ_2, ψ satisfy (52)-(54). We then note that (73)-(74) and the relationship between the triple cross products on $\mathbb{R}^7$ and $\mathbb{R}^8$ ensure that (75)-(77) are equivalent to (52)-(54). The result follows. $\square$

4.4 Holomorphic Vector Fields

We now finish this section by giving a means of constructing r-framed 2-ruled calibrated 4-folds starting from r-framed 2-ruled calibrated cones using *holomorphic vector fields*, which is analogous to [5, Theorem 6.1].

Suppose that M_0 is an r-framed, 2-ruled, Cayley cone in $\mathbb{R}^8$ defined by maps $\phi_1, \phi_2 : \Sigma \to \mathbb{R}^8$ as in (9). Then Proposition 4.7 gives us a conformal structure on Σ related to ϕ_1, ϕ_2 and hence we can consider Σ as a *Riemann surface*. Therefore Σ has a natural complex structure J and we may define oriented conformal coordinates (s,t) on Σ. Suppose further that ϕ_1, ϕ_2 are in the flat gauge. Hence the equations ϕ_1, ϕ_2 satisfy are:

$$\frac{\partial \phi_j}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial s} \qquad \text{for } j = 1, 2, \tag{78}$$

$$\frac{\partial \phi_j}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial t} \qquad \text{for } j = 1, 2. \tag{79}$$

We note from equations (78) and (79) that there is a correspondence between "$\phi_1 \times \phi_2 \times$" and the complex structure J on Σ.

Theorem 4.10 *Let M_0 be an r-framed, 2-ruled, Cayley cone in $\mathbb{R}^8$ defined by maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define a map $\psi : \Sigma \to \mathbb{R}^8$ by $\psi = \mathcal{L}_w \phi_1 + \mathcal{L}_{iw} \phi_2$, where $\mathcal{L}_w, \mathcal{L}_{iw}$ denote the Lie derivatives with respect to w, iw. Let M be defined by ϕ_1, ϕ_2, ψ as in* (51). *Then M is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$.*

Proof: We need to show that ψ as defined satisfies (54). If w is identically zero, then ψ trivially satisfies (54). Therefore we need only consider the case where w has isolated zeros. Since the condition for M to be Cayley is a closed condition on M, it is sufficient to prove that (54) holds at any point $\sigma \in \Sigma$ with $w(\sigma) \neq 0$.

Let $\sigma \in \Sigma$ be such a point. Then, since w is a holomorphic vector field, there exists an open set in Σ containing σ with oriented conformal coordinates (s,t) such that $w = \frac{\partial}{\partial s}$, so that $iw = \frac{\partial}{\partial t}$. Hence $\psi = \frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_2}{\partial t}$ in a neighbourhood of σ.

Let $(e_1, \ldots, e_8)$ be an oriented orthonormal basis for $\mathbb{R}^8$ and $A = |\frac{\partial \phi_1}{\partial s}(\sigma)|$. We transform coordinates on $\mathbb{R}^8$ using Spin(7) such that $\phi_1(\sigma) = e_1$, $\phi_2(\sigma) = e_2$, $\frac{\partial \phi_1}{\partial s}(\sigma) = Ae_3$ and $\frac{\partial \phi_2}{\partial s}(\sigma) = a_1 e_1 + \ldots + a_8 e_8$, for some real constants $a_1, \ldots, a_8$. Clearly, by (78), $\frac{\partial \phi_1}{\partial t}(\sigma) = Ae_4$ and hence $a_1 = a_2 = a_4 = 0$ by the orthogonality conditions imposed on $\frac{\partial \phi_2}{\partial s}$ in the flat gauge. Differentiating (78) gives:

$$
\begin{aligned}
\frac{\partial^2 \phi_1}{\partial s \partial t} &= \phi_1 \times \frac{\partial \phi_2}{\partial s} \times \frac{\partial \phi_1}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2 \phi_1}{\partial s^2}, \\
\frac{\partial^2 \phi_2}{\partial t^2} &= \frac{\partial \phi_1}{\partial t} \times \phi_2 \times \frac{\partial \phi_2}{\partial s} + \phi_1 \times \frac{\partial \phi_2}{\partial t} \times \frac{\partial \phi_2}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2 \phi_2}{\partial t \partial s}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_2}{\partial t}\right) \\
&= \phi_1 \times \frac{\partial \phi_2}{\partial s} \times \frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_1}{\partial t} \times \phi_2 \times \frac{\partial \phi_2}{\partial s} + \phi_1 \times \frac{\partial \phi_2}{\partial t} \times \frac{\partial \phi_2}{\partial s} \\
&\quad + \phi_1 \times \phi_2 \times \frac{\partial}{\partial s}\left(\frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_2}{\partial t}\right).
\end{aligned}
$$

Calculation using (19) and (78) shows that

$$
\begin{aligned}
\phi_1 \times \frac{\partial \phi_2}{\partial s} \times \frac{\partial \phi_1}{\partial s}(\sigma) &= A(a_7 e_5 - a_8 e_6 - a_5 e_7 + a_6 e_8), \\
\frac{\partial \phi_1}{\partial t} \times \phi_2 \times \frac{\partial \phi_2}{\partial s}(\sigma) &= A(-a_3 e_1 - a_7 e_5 + a_8 e_6 + a_5 e_7 - a_6 e_8), \\
\phi_1 \times \frac{\partial \phi_2}{\partial t} \times \frac{\partial \phi_2}{\partial s}(\sigma) &= -(a_3^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2) e_2.
\end{aligned}
$$

We conclude that ψ satisfies (54) at σ with $g_1(\sigma) = -Aa_3 = -g(\frac{\partial \phi_1}{\partial s}(\sigma), \frac{\partial \phi_2}{\partial s}(\sigma))$ and $g_2(\sigma) = -|\frac{\partial \phi_2}{\partial s}(\sigma)|^2$. By the invariance of the metric and the triple cross product under Spin(7), and the discussion above, ψ satisfies (54) for some $g_1, g_2 : \Sigma \to \mathbb{R}$. Hence, using Theorem 4.6, the result follows. $\square$

This result does not extend to the SL case in the way we might expect. The construction starting with a 2-ruled SL cone M_0 will generally produce a 2-ruled Cayley, but not SL, 4-fold M. The fact that M is Cayley follows trivially from Theorem 4.10, but if we impose the condition $\omega|_M \equiv 0$, then ϕ_1, ϕ_2 must satisfy

$$
\begin{aligned}
\omega\left(\frac{\partial \phi_1}{\partial s}, \frac{\partial \phi_2}{\partial s}\right) &= -\omega\left(\frac{\partial \phi_1}{\partial t}, \frac{\partial \phi_2}{\partial t}\right) = 0, \\
\omega\left(\frac{\partial \phi_1}{\partial s}, \frac{\partial \phi_2}{\partial t}\right) &= \quad \omega\left(\frac{\partial \phi_1}{\partial t}, \frac{\partial \phi_2}{\partial s}\right) = 0,
\end{aligned}
$$

wherever $w \neq 0$. At such a point, either all the derivatives of ϕ_1, ϕ_2 are zero or at least one is non-zero. In the first case both ϕ_1 and ϕ_2 are locally constant.

Otherwise, suppose without loss of generality that $\frac{\partial\phi_1}{\partial s} \neq 0$ at a point σ such that $w(\sigma) \neq 0$. Then $\langle \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t} \rangle_{\mathbb{C}} = \mathbb{C}^2$, and it is orthogonal to $\langle \phi_1, \phi_2 \rangle_{\mathbb{C}} = \mathbb{C}^2$ since M is SL and ϕ_1, ϕ_2 are in the flat gauge. Therefore $\frac{\partial\phi_2}{\partial s} \in \langle \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t} \rangle_{\mathbb{C}}$. Note that $g(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}) = \omega(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}) = 0$ and $\omega(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}) = 0$. Hence there exists $\theta \in \mathbb{R}$ such that $\cos\theta\frac{\partial\phi_1}{\partial s} + \sin\theta\frac{\partial\phi_2}{\partial s} = 0$. Using (78), $\cos\theta\frac{\partial\phi_1}{\partial t} + \sin\theta\frac{\partial\phi_2}{\partial t} = 0$. Therefore, $\cos\theta\phi_1 + \sin\theta\phi_2$ is constant on a neighbourhood of σ and thus on the component of Σ containing σ. Hence we have the following result.

Theorem 4.11 *Let M_0 be an r-framed, 2-ruled, SL cone in $\mathbb{C}^4 \cong \mathbb{R}^8$ defined by $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define $\psi : \Sigma \to \mathbb{C}^4$ by $\psi = \mathcal{L}_w\phi_1 + \mathcal{L}_{iw}\phi_2$, where $\mathcal{L}_w, \mathcal{L}_{iw}$ denote the Lie derivatives with respect to w, iw. Let M be defined by ϕ_1, ϕ_2, ψ as in* (59)*. Then M is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$, which is SL if and only if $w \equiv 0$ or there exists $\theta \in \mathbb{R}$ for each component K of Σ such that $\cos\theta\phi_1 + \sin\theta\phi_2$ is constant on K.*

We do, however, have a similar result to Theorem 4.10 for coassociative 4-folds.

Theorem 4.12 *Let M_0 be an r-framed, 2-ruled, coassociative cone in $\mathbb{R}^7$ defined by $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^6$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define $\psi : \Sigma \to \mathbb{R}^7$ by $\psi = \mathcal{L}_w\phi_1 + \mathcal{L}_{iw}\phi_2$, where $\mathcal{L}_w, \mathcal{L}_{iw}$ denote the Lie derivatives with respect to w, iw. Let M be defined by ϕ_1, ϕ_2, ψ as in* (72)*. Then M is an r-framed 2-ruled coassociative 4-fold in $\mathbb{R}^7$.*

Proof: This follows immediately from Theorem 4.10 since $M \subseteq \mathbb{R}^7 \subseteq \mathbb{R}^8$ is Cayley and therefore coassociative by Proposition 2.11. □

5 Examples

We shall now exhibit explicit examples of 2-ruled 4-folds.

5.1 U(1)-Invariant 2-ruled Cayley 4-folds

We consider the family of SL 4-folds in $\mathbb{C}^4$ given in [3, Theorem III.3.1]. Let $\mathbf{c} = (c_1, c_2, c_3, c_4) \in \mathbb{R}^4$ be constant and define $M_{\mathbf{c}} \subseteq \mathbb{C}^4$ by:

$$M_{\mathbf{c}} = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \mathrm{Re}(z_1z_2z_3z_4) = c_1,\ |z_1|^2 - |z_j|^2 = c_j \text{ for } j = 2, 3, 4\}. \tag{80}$$

Then $M_{\mathbf{c}}$ is an SL 4-fold in $\mathbb{C}^4$ invariant under $\mathrm{U}(1)^3$.

Taking $\mathbf{c} = 0$, we see that M_0 is an r-framed 2-ruled SL cone in $\mathbb{C}^4$, with three different 2-rulings. For each of the distinct 2-rulings we are then able to apply the holomorphic vector field result of Theorem 4.11 to obtain families of r-framed 2-ruled Cayley 4-folds which are invariant under U(1).

Theorem 5.1 *Let $w : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then*

$$M_1 = \Big\{ \frac{1}{2}\Big(ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right), \\ e^{it}\left(re^{-i\theta} + w(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right) \Big) : r,s,t,\theta \in \mathbb{R} \Big\}, \quad (81)$$

$$M_2 = \Big\{ \frac{1}{2}\Big(ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right), \\ e^{it}\left(re^{-i\theta} + w(s+it)\right), e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right) \Big) : r,s,t,\theta \in \mathbb{R} \Big\}, \quad (82)$$

$$M_3 = \Big\{ \frac{1}{2}\Big(ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{it}\left(re^{-i\theta} + w(s+it)\right), \\ e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right) \Big) : r,s,t,\theta \in \mathbb{R} \Big\} \quad (83)$$

are r-framed 2-ruled Cayley 4-folds in $\mathbb{R}^8 \cong \mathbb{C}^4$.

Proof: We only prove the result for M_1 as the proof for the other two is similar. In this example we define M_0 by functions $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathcal{S}^7 \subseteq \mathbb{C}^4$ given by:

$$\phi_1(s,t) = \frac{1}{2}(ie^{is}, e^{-is}, e^{it}, e^{-it}), \quad (84)$$

$$\phi_2(s,t) = \frac{i}{2}(ie^{is}, e^{-is}, -e^{it}, -e^{-it}), \quad (85)$$

so that M_0 is 2-ruled by planes of the form:

$$\Pi_{r,\theta} = \{r\cos\theta\phi_1(s,t) + r\sin\theta\phi_2(s,t) : s,t \in \mathbb{R}\} \\ = \left\{ \frac{r}{2}\left(ie^{i(\theta+s)}, e^{i(\theta-s)}, e^{-i(\theta-t)}, e^{-i(\theta+t)}\right) : s,t \in \mathbb{R} \right\}.$$

We verify through direct calculation that $g(\phi_1,\phi_2) = \omega(\phi_1,\phi_2) = 0$, (60), (62) and (63) are satisfied for $f = 0$, and that ϕ_1, ϕ_2 are in the flat gauge.

Let $w(s+it) = u(s,t) + iv(s,t)$ for functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then $\psi = \mathcal{L}_w\phi_1 + \mathcal{L}_{iw}\phi_2$ is given by:

$$\psi(s,t) = u(s,t)\frac{\partial\phi_1}{\partial s} + v(s,t)\frac{\partial\phi_1}{\partial t} - v(s,t)\frac{\partial\phi_2}{\partial s} + u(s,t)\frac{\partial\phi_2}{\partial t} \\ = \frac{1}{2}\left(i\bar{w}(s+it)ie^{is}, -i\bar{w}(s+it)e^{-is}, w(s+it)e^{it}, -w(s+it)e^{-it}\right). \quad (86)$$

We check that ψ satisfies (54) of Theorem 4.6 with $g_1 = 0$ and $g_2 = -\frac{1}{2}$, which agrees with the calculations in the proof of Theorem 4.10. Therefore, by Theorem 4.10, M_1 as defined in (81) is an r-framed 2-ruled Cayley 4-fold. $\square$

Note, from the proof above, that $2\omega(\phi_1, \frac{\partial\psi}{\partial s}) = v$ and $2\omega(\phi_1, \frac{\partial\psi}{\partial s}) = u$, so that (61) of Theorem 4.8 is satisfied if and only if $w \equiv 0$. Therefore, if w is not identically zero, Theorem 4.11 shows that M_1 is an r-framed 2-ruled Cayley 4-fold which is not special Lagrangian. Similarly for M_2 and M_3.

An interesting special case is when w in Theorem 5.1 is taken to be constant. Here, calculation shows that, each M_j is invariant under a $\mathrm{U}(1)^2$ subgroup of $\mathrm{U}(1)^3$. Moreover, they are asymptotically conical to M_0 with order $O(r^{-1})$, in the sense of Definition 3.2.

5.2 Ruled Associative and Special Lagrangian 3-folds

We can construct examples of 2-ruled 4-folds from *ruled* associative 3-folds in $\mathbb{R}^7$ and SL 3-folds in $\mathbb{C}^3$, as described in [6, §6] and [5]. We first give the definitions, of which Definition 3.1 is the analogue for 2-ruled 4-folds.

Definition 5.2 Let M be a 3-dimensional submanifold of $\mathbb{R}^7$ (or $\mathbb{C}^3$). A *ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that for all $\sigma \in \Sigma$ there exist $\mathbf{v}_\sigma \in \mathcal{S}^6$ (or $\mathcal{S}^5$), $\mathbf{w}_\sigma \in \mathbb{R}^7$ (or $\mathbb{C}^3$) such that $\pi^{-1}(\sigma) = \{r\mathbf{v}_\sigma + \mathbf{w}_\sigma : r \in \mathbb{R}\}$. Then the triple (M, Σ, π) is a *ruled 3-fold* in $\mathbb{R}^7$ (or $\mathbb{C}^3$).

An *r-orientation* for a ruling (Σ, π) is a choice of orientation for the affine straight line $\pi^{-1}(\sigma)$ in $\mathbb{R}^7$ (or $\mathbb{C}^3$), for each $\sigma \in \Sigma$, which varies continuously with σ. Then a ruled 3-fold (M, Σ, π) with an r-orientation is called an *r-oriented ruled 3-fold*.

Let (M, Σ, π) be an r-oriented ruled 3-fold. For each $\sigma \in \Sigma$ define $\phi(\sigma)$ to be the unique unit vector in $\mathcal{S}^6$ (or $\mathcal{S}^5$) parallel to $\pi^{-1}(\sigma)$ and in the positive direction with respect to the orientation on $\pi^{-1}(\sigma)$, given by the r-orientation. Then $\phi : \Sigma \to \mathcal{S}^6$ (or $\mathcal{S}^5$) is a smooth map. Define $\psi : \Sigma \to \mathbb{R}^7$ (or $\mathbb{C}^3$) such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi(\sigma)$. Then ψ is smooth and

$$M = \{r\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\, r \in \mathbb{R}\}. \tag{87}$$

In the ruled case, there is a natural way to define a metric on Σ as the pullback $\phi^*(g)$ of the round metric g on $\mathcal{S}^6$ (or $\mathcal{S}^5$), and hence we may always

define oriented conformal coordinates in terms of a natural complex structure on Σ.

Suppose that (N, Σ, π) is a ruled 3-fold. Let $M = \mathbb{R} \times N$ and let $\tilde{\pi} : M \to \Sigma$ be given by $\tilde{\pi}(r, p) = \pi(p)$ for all $p \in N$. Clearly, $(M, \Sigma, \tilde{\pi})$ is a 2-ruled 4-fold since $\tilde{\pi}^{-1}(\sigma) = \mathbb{R} \times \pi^{-1}(\sigma)$ for all $\sigma \in \Sigma$. Suppose further that (N, Σ, π) is r-oriented. Using the r-orientation, we have a natural choice of oriented orthonormal basis for the plane $\tilde{\pi}^{-1}(\sigma)$, which varies smoothly with σ. Therefore, $(M, \Sigma, \tilde{\pi})$ is r-framed.

We now state and prove the following theorem.

Theorem 5.3 (a) *Let $N \subseteq \mathbb{R}^7$ be an (r-oriented) ruled associative 3-fold. Then $\mathbb{R} \times N \subseteq \mathbb{R} \oplus \mathbb{R}^7 \cong \mathbb{R}^8$ is an (r-framed) 2-ruled Cayley 4-fold.*

(b) *Let $L \subseteq \mathbb{C}^3$ be an (r-oriented) SL 3-fold with phase $-i$. Then $\mathbb{R} \times L \subseteq \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ is an (r-framed) 2-ruled coassociative 4-fold.*

Proof: Let N be an associative 3-fold in $\mathbb{R}^7$. Then, by equation (6), since $*\varphi|_{\mathbb{R}\times N} \equiv 0$ and N is a φ-submanifold, it is clear that $\mathbb{R} \times N$ is calibrated with respect to Φ. The comments before the theorem then give the result (a).

Let L be an SL 3-fold with phase $-i$ in $\mathbb{C}^3$. Therefore, L is calibrated with respect to $-\operatorname{Im}\Omega$, where Ω is the holomorphic volume form on $\mathbb{C}^3$. Let (z_1, z_2, z_3) be complex coordinates on $\mathbb{C}^3$. We identify $\mathbb{R}\oplus\mathbb{C}^3$ and $\mathbb{R}^7$ by defining coordinates on $\mathbb{R}^7$ as $(x_1, \dots, x_7)$ where x_1 is the coordinate on $\mathbb{R}$ and we let $z_1 = x_2 + ix_3$, $z_2 = x_4 + ix_5$, $z_3 = x_6 + ix_7$. Then,

$$*\varphi = \frac{1}{2}\,\omega \wedge \omega - dx_1 \wedge \operatorname{Im}\Omega. \tag{88}$$

Clearly, $\omega \wedge \omega|_{\mathbb{R}\times L} \equiv 0$ as $\omega|_L \equiv 0$ and hence $\mathbb{R} \times L$ is coassociative. Again, we use the results before the theorem to give the result (b). □

5.3 Complex Cones

We define a complex cone C in $\mathbb{C}^4$ by

$$C = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : P(z_1, z_2, z_3, z_4) = Q(z_1, z_2, z_3, z_4) = 0\}$$

where P, Q are homogeneous complex polynomials. Suppose further that P, Q are such that C is non-planar and nonsingular except at 0. Define a projection $\tilde{\pi}$ from $C \setminus 0$ to $\mathbb{CP}^3$ by $\tilde{\pi}((z_1, z_2, z_3, z_4)) = [z_1, z_2, z_3, z_4]$ and let Σ be the image of $\tilde{\pi}$. Let M_0 be given by

$$M_0 = \{(z_1, z_2, z_3, z_4, \sigma) \in C \times \Sigma : (z_1, z_2, z_3, z_4) \in \sigma\}$$

and define $\iota : M_0 \to \mathbb{C}^4$ by $\iota(z_1, z_2, z_3, z_4, \sigma) = (z_1, z_2, z_3, z_4)$. Then ι is an immersion except at 0 and thus M_0 can be considered as an immersed submanifold of $\mathbb{C}^4$ which is only singular at 0. Let $\pi : M_0 \to \Sigma$ be given by $\pi(z_1, z_2, z_3, z_4, \sigma) = \sigma$. Clearly, M_0 is 2-ruled by complex lines $\pi^{-1}(\sigma)$ in $\mathbb{C}^4$. Since any complex surface in $\mathbb{C}^4 \cong \mathbb{R}^8$ is Cayley by [3, §IV.2.C], (M_0, Σ, π) is a 2-ruled Cayley 4-fold.

We can define a local holomorphic coordinate w, and hence oriented conformal coordinates (s, t), in Σ by $w \mapsto [z_1, z_2, z_3, z_4](w)$ in some open set U in Σ. Suppose without loss of generality that $z_4 \neq 0$ in U. Then, we may rescale so that $z_4 = 1$ and define maps $\phi_1, \phi_2 : U \to \mathcal{S}^7$ by

$$\begin{aligned} \phi_1(s,t) &= \left(\frac{z_1(s,t)}{r}, \frac{z_2(s,t)}{r}, \frac{z_3(s,t)}{r}, \frac{1}{r} \right), \\ \phi_2(s,t) &= i\phi_1(s,t), \end{aligned}$$

where $r = (1 + |z_1(s,t)|^2 + |z_2(s,t)|^2 + |z_3(s,t)|^2)^{\frac{1}{2}}$. We can then write M_0 locally in the form (9), where ϕ_1, ϕ_2 satisfy (52)-(53), since C, and hence M_0, is non-planar. If we define M by (51) where ψ satisfies (54) then, from Theorem 4.6, M is a non-planar, r-framed, 2-ruled Cayley 4-fold in $\mathbb{R}^8$.

A Appendix

A.1 Cayley Multiplication Table for the Octonions

Let $e_1 = 1$ and let $\{e_2, \dots, e_8\}$ be a basis for Im $\mathbb{O}$. Then a Cayley multiplication table for the octonions is as shown:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	e_8	$-e_7$
e_3	e_3	$-e_4$	$-e_1$	e_2	e_7	$-e_8$	$-e_5$	e_6
e_4	e_4	e_3	$-e_2$	$-e_1$	$-e_8$	$-e_7$	e_6	e_5
e_5	e_5	$-e_6$	$-e_7$	e_8	$-e_1$	e_2	e_3	$-e_4$
e_6	e_6	e_5	e_8	e_7	$-e_2$	$-e_1$	$-e_4$	$-e_3$
e_7	e_7	$-e_8$	e_5	$-e_6$	$-e_3$	e_4	$-e_1$	e_2
e_8	e_8	e_7	$-e_6$	$-e_5$	e_4	e_3	$-e_2$	$-e_1$

Note that the multiplication is defined above so as to be compatible with the formula (1) for φ and hence the table is not the standard one.

A.2 Calculating the Fourfold Cross Product

At various stages in the proof of Theorem 4.2 we are required to make calculations involving the fourfold cross product on $\mathbb{O} \cong \mathbb{R}^8$. We shall give here details of these calculations and the methods employed in order to compute these products efficiently.

For our purposes we need only consider fourfold cross products of the form

$$f_{jk} = e_1 \times e_2 \times e_j \times e_k.$$

It is clear by Definition 2.7 that f_{jk} is antisymmetric and that $f_{jk} = 0$ for $1 \leq j, k \leq 2$ since the fourfold cross product is alternating.

We only want to consider the case when $\operatorname{Im} f_{jk} \neq 0$. By Proposition 2.8 this occurs if and only if $\{e_1, e_2, e_j, e_k\}$ does not lie in a Cayley 4-plane. Hence we deduce that $\operatorname{Im} f_{jk} = 0$ if $\{j,k\} = \{3,4\}$ or $\{j,k\} = \{5,6\}$ or $\{j,k\} = \{7,8\}$.

We next make the following observation. By the invariance of the fourfold cross product under Spin(7), if $\{e_j, e_k, e_l, e_m\}$ is an ordered basis for a Cayley 4-plane, then either $f_{jk} = f_{lm}$ or $f_{jk} = -f_{lm}$, depending on whether $\{e_j, e_k, e_l, e_m\}$ is a positively oriented basis or not.

Therefore, the only fourfold cross products we require are:

$$\begin{aligned} f_{58} &= e_3 = f_{67}, \\ f_{57} &= e_4 = f_{86}, \\ f_{74} &= e_5 = f_{83}, \\ f_{48} &= e_6 = f_{73}, \\ f_{36} &= e_7 = f_{45}, \\ f_{35} &= e_8 = f_{64}. \end{aligned}$$

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